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THE UNIVERSITY OF WOLLONGONG
DEPARTMENT OF ECONOMICS

RAY SCALE ECONOMIES AND MULTIPRODUCT COST
FUNCTIONS

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ABSTRACT

The purpose of this paper is to show that if the ray scale economies depend on the output vector alone, it is possible to derive the full behaviour of costs as output bundles change. The formal proofs provide operational criteria for generating multiproduct cost functions for which ray scale economies depend either on cost level alone, or on output proportions alone, in a presassigned manner.

1. Introduction

In the multiproduct case average cost cannot be defined, and therefore the standard definition of scale economies cannot be applied. An analytically tractable concept of scale economies in a multiproduct setting -introduced by Baumol (1977), and Panzar and Willig (1977)- is termed "ray scale economies". It is a straightforward extension of the concept of single-product scale economies and indicates the behavior of costs as the production levels of a given bundle of outputs change equiproportionately. Since ray economies of scale may be different at different scales of operation, it may be useful to characterize the classes of cost functions for which economies of scale can be described in terms of the output vector alone. The purpose of this paper is to show that a knowledge of a ray scale economies function having this property allows the derivation of the multiproduct cost function.

2. Directional elasticity and ray average cost

Let $C: Y \rightarrow R$ be a multiproduct cost function defined by the formula $C = C(y) = C(y_1, \dots, y_n)$, where y_i ($i=1, \dots, n$) stands for the physical quantity of output i , Y for the nonnegative orthant of R^n , and R for the set of nonnegative real numbers. By *directional elasticity of the total cost function* $C = C(y)$ evaluated at and in the direction of the output vector y it is meant

$$a(y) = \frac{y}{C(y)} \lim_{\theta \rightarrow 1} \frac{C(\theta y) - C(y)}{\theta - 1}, \quad \theta > 0, \quad (1)$$

if the limit exists. When $C(y)$ is differentiable at y , the limit (1) exists and it turns out to be the sum of all partial cost elasticities, i.e.

$$a(y) = \frac{1}{C(y)} \sum_{i=1}^n y_i \frac{\partial C(y)}{\partial y_i} \quad (2)$$

Ray average cost (RAC) is the cost of an output vector of fixed proportions divided by an homogeneous measure of the size of the outputs. Analytically, the RAC in terms of the measure θ evaluated at the output vector y is

$$RAC = \frac{C(\theta y)}{\theta}, \quad \theta > 0 \quad (3)$$

It is not difficult to show that the numerical value of $a(y)$ determines the sign of the slope of the RAC. In fact, the differentiation of (3) with respect to θ , followed by setting $\theta = 1$, yields

$$\frac{dRAC(y)}{d\theta} = a(y) - 1 \quad (4)$$

In particular, a sufficient condition of strictly decreasing RAC is that the directional elasticity be less than unity [Cf. Baumol et al. (1982, p.51)]. From now on, $a(y)$ will be referred to as a *ray scale economies function*.

3. Homotheticity and ray scale economies functions

Before going into the proof of an operational method for generating multiproduct cost functions with variable scale economies some additional nomenclature is in order. Let H denote the class of all positively homogeneous functions of degree m . Let $f \in H$ be a multiproduct cost function. A

continuously differentiable function $C = g(f)$, where $g(0)=0$ and $g'(f)>0$ for $0<f<\infty$, is termed a *homothetic cost function* (HCF). Finally, to make explicit contact with scale economies in a multiproduct setting it is necessary to indicate a stronger definition of $a(y)$. Let $C = C(y)$ be an HCF and consider a continuously differentiable function $h: R \rightarrow R$. The composite function $a = h \circ g \circ f: Y \rightarrow R$ is a *ray scale economies function*, if

$$y \nabla C(y) = a(y)C(y) \quad (5)$$

Given a homogeneous cost function and a ray scale economies function it is shown below how these functions can be transformed to yield an HCF exhibiting the preassigned scale economies.

Theorem 1

Let $f \in H$. There exists an HCF, $C = C(y)$, with preassigned scale economies function $a = a(y)$, if and only if, C satisfies the following differential equation:

$$\frac{d \log C}{d \log f} = \frac{h(C)}{m} \quad (6)$$

Proof

Necessity. If there exists an HCF, $C = C(y)$, with $h(C)$ preassigned,

$$y \nabla C(y) = y \frac{dC}{df} \nabla f = h(C)C \quad (7)$$

But since $f \in H$, it follows from Euler's theorem that

$$\frac{dC}{df} m f = h(C)C, \quad (8)$$

i.e. the desired result.

Sufficiency. The general solution of equation (6) can be written as:

$$\exp[\phi(y)] = kf \quad (9)$$

where

$$\phi(y) = m \int dC/h(C)C \quad (10)$$

and k is a constant of integration. It is shown in the Appendix that (9) is the general solution of the partial differential equation (5). Thus sufficiency is proved.

4. Inhomothetic cost functions.

HCFs have scale economies invariant to y along isocost contours, so that when $a(y)$ is homogeneous of degree zero, the preceding theorem breaks down. In fact, a cost function generating constant a along each ray, but different values of a along every ray, cannot be an HCF. Those cost functions for which ray scale economies depend on output proportions alone will be termed *inhomothetic cost functions (ICF)*.

Theorem 2

The functional form of an ICF is

$$C(y) = [B(y)]^{a(y)} \quad (11)$$

where $B(y)$ is an arbitrary function homogeneous of degree one, and $a(y)$ is homogeneous of degree zero.

Proof

Using the same technique as in the Appendix, the general solution of (5) is readily seen to be given by

$$F(K_1, \dots, K_n) = 0, \quad (12)$$

where $F(\cdot)$ is an arbitrary function,

$$K_{i-1} = \frac{y_i}{y_1} \quad (i=2, \dots, n) \quad (13)$$

and

$$K_n = C [y_1^{a(y)}]^{-1} \quad (14)$$

Equation (12) must be soluble as

$$K_n = [J(K_1, \dots, K_{n-1})]^{a(y)} \quad (15)$$

where $J(\cdot)$ is an arbitrary function taking positive values for $K_{i-1} > 0$, $i=2, \dots, n$. Therefore,

$$C = [J(K_1, \dots, K_{n-1}) y_1]^{a(y)} \quad (16)$$

and the proof is complete.

5. Conclusions

A duality for the multiproduct firm exists between the ray scale economies and the cost functions. Theorem 1 and 2 provide operational criteria for generating multiproduct cost functions with variable scale economies. In fact, a multiproduct HCF associated with a particular choice of the ray scale economies function can be obtained by solving the differential equation of variables separated form (6). Moreover, if the ray scale economies depend on the output proportions alone, the class of ICF is described by (11).

Several methods for measuring scale economies have been developed since the 1930s. The message that comes across with this paper is that an econometric estimation of the parameters of the ray scale economies function permits the derivation of the underlying multiproduct cost function.

Appendix

The general solution of the quasi-linear first order partial differential equation (5) is given by $\exp[\phi(y)] = kf$, where k is a constant, $\phi(y) = \int dC/h(C)C$, and $f \in H$ is an arbitrary function.

Proof

The technique of solution employed here is discussed in Smirnov (1964, esp. pp.310-316). The differential system

$$dy_1/y_1 = \dots = dy_n/y_n = dC/h(C)C \quad (18)$$

is equivalent to equation (5). Clearly, the first integrals

$$y_2/y_1 = k_1, \dots, y_n/y_1 = k_{n-1}, \exp[\phi(y)]/y_1^m = k_n \quad (19)$$

are independent. Then system (18) has a general solution of the form

$$G\{y_2/y_1, \dots, y_n/y_1, \exp[\phi(y)]/y_1^m\} = 0, \quad (20)$$

where $G\{\}$ is an arbitrary function. Solving equation (20) for the last argument, yields:

$$\exp[\phi(y)] = y_1^m J(y_2/y_1, \dots, y_n/y_1) \quad (21)$$

Thus, $J \in H$. Let $J = kf$, where k is a constant and $f \in H$, and the proof is complete.

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